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New Twistorial Integral Formulas for Massless Free Fields of Arbitrary Spin

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A manifestly scaling-invariant version of the Kirchoff–D'Adhemar–Penrose field integrals is presented. The invariant integral expressions for the spinning massless free fields are directly transcribed into the framework of twistor theory. It is then shown that the resulting twistorial field integrals can be thought of as being equivalent to the universal Penrose contour integral formulas for these fields.

1. INTRODUCTION

The Kirchoff–D'Adhemar–Penrose (KAP) integral expressions can be looked upon as powerful tools for evaluating spinning massless free fields on real Minkowski space in an explicit way (Penrose, 1963, 1980). In these expressions, the null initial data (NID) for the fields are specified at arbitrary nonsingular points of real null hypersurfaces. Those components of the fields which are associated with the (null) directions of the generators of the hypersurfaces are, effectively, regarded as the NID for the fields. Particularly, the field integrals involve the derivatives of the NID for the fields in the directions of the generators of the NID hypersurfaces. Indeed, these directional derivatives, being defined at nonsingular points of the NID hypersurfaces, suitably combine the usually required normal and tangential derivatives of the NID for the fields (Penrose and Rindler, 1984; Penrose, 1975). Moreover, the NID for the fields enter into the field integrals together with the convergence of the generators of the NID hypersurfaces. The field integrals are taken over the (smooth) cross sections which are provided by the intersection of appropriate null cones with the NID hypersurfaces.

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This paper is mainly concerned with transcribing a slightly modified version of the KAP field integrals into the framework of twistor theory. We will restrict ourselves to considering the future null cone of an origin of real Minkowski space as the NID hypersurface for all the fields. In particular, it will enable us to carry out the entire twistorial transcription in a straightforward way (Section 3). Our modified version of the field integrals is manifestly invariant under arbitrary scalings of the elements of certain spinor bases. These are particularly used for defining the relevant NID sets for the fields (Section 2). The modified field integrals are taken over the two-dimensional space of all null zigzags starting at the origin of real Minkowski space and terminating at a fixed point of the interior of the future cone of the origin. Indeed, such a point defines a future-timelike vector in real Minkowski space. Actually, the introduction of these integrals is based upon a particularly simple Huyghens' principle. The standard twistor functions representing the spinning massless free fields (see, for example, Penrose and Ward, 1980) arise here as suitable contour integrals of simple holomorphic twistor one-forms. This fact will enable us to derive the universal Penrose contour integral formulas for the spinor fields out of our twistorial field integrals (Section 4). It may well be said that the main motivation for this work is the fact that one might eventually gain new insights into the theory of twistors upon transcribing suitably modified expressions for massless free fields of arbitrary spin.

The unprimed and primed spinor fields with which we deal here will be referred to as left-handed and right-handed fields, respectively. However, there will be no attempt to consider these fields as quantum fields at this stage. We make use of it as conventional terminology, and even use it for designating quantities other than the spinor fields. In addition, it will be effectively assumed that the spinor fields are analytic functions throughout real Minkowski space.

2. BASIC DEFINITIONS AND FIELD INTEGRALS

The main aim of this section is to exhibit the invariant version of the integral expressions for the spinning massless free fields. We shall first introduce some relevant basic definitions and the NID sets that play an important role here. The explicit field integrals are then presented.

2.1. Massless Free Fields on Real Minkowski Space

A left-handed massless free field of spin s on real Minkowski space $\mathbb{RM}$ is a totally symmetric spinor field $\phi_{AB\cdots C}(x)$ with $n=-2s$ unprimed indices, which satisfies the massless free-field equation

$$\nabla^{AA'}\phi_{AB\cdots C}(x)=0 \qquad (2.1.1)$$

throughout $\mathbb{RM}$. It can be regarded as a complex-valued function, of the usual four real variables x^0, x^1, x^2, x^3, on $\mathbb{RM}$.

Similarly, a right-handed massless free field of spin s on $\mathbb{RM}$ is a corresponding totally symmetric spinor field $\theta^{A'B'\cdots C'}(x)$ with $n=2s$ primed indices, which satisfies the complex conjugate form of (2.1.1)

$$\nabla_{AA'}\theta^{A'B'\cdots C'}(x)=0 \tag{2.1.2}$$

It is easily verified that the wave equation

$$\nabla^{AA'}\nabla_{AA'}\vartheta(x)=\Box\vartheta(x)=0 \tag{2.1.3}$$

holds throughout $\mathbb{RM}$, with $\vartheta(x)$ standing for any component of either $\phi_{AB\cdots C}(x)$ or $\theta^{A'B'\cdots C'}(x)$.

2.2. Null Initial Data Sets

We are now in a position to introduce a prescription for defining the relevant NIDSs. Some definitions of weighted scalars that are considered in the spin-coefficient formalisms will be used here (see Penrose and Rindler, 1984).

The future null cone C_0^+ of an origin O of $\mathbb{RM}$ is a three-dimensional manifold in $\mathbb{RM}$, given by

$$C_0^+=\{\text{future-pointing vectors } x^{AA'}\in\mathbb{RM}\,|\,\varepsilon_{AB}\varepsilon_{A'B'}x^{AA'}x^{BB'}=0\} \tag{2.2.1}$$

Clearly, the set of singular points of C_0^+ consists of the origin. Let $\overset{1}{x}{}^{AA'}$ be some nonsingular point of C_0^+. Set up a pair of spin bases

$$\{\{\overset{1}{o}{}^A,\overset{2}{o}{}^A\},\{\underset{1}{\bar{o}}{}^{A'},\underset{2}{\bar{o}}{}^{A'}\}\}\quad\text{at}\quad\overset{1}{x}{}^{AA'}$$

such that the real null vectors

$$\{\overset{1}{o}{}^A\underset{1}{\bar{o}}{}^{A'},\overset{2}{o}{}^A\underset{2}{\bar{o}}{}^{A'}\} \tag{2.2.2}$$

point in forward (future) null directions through $\overset{1}{x}{}^{AA'}$. Also, define the inner products at $\overset{1}{x}{}^{AA'}$

$$\overset{1}{z}=\overset{1}{o}{}^A\overset{2}{o}_A,\qquad \underset{1}{\bar{z}}=\underset{1}{\bar{o}}{}^{A'}\underset{2}{\bar{o}}_{A'} \tag{2.2.3}$$

and let the spinors $\overset{1}{o}{}^A$ and $\underset{1}{\bar{o}}{}^{A'}$ be chosen covariantly constant along the generator γ_1 of C_0^+ that passes through $\overset{1}{x}{}^{AA'}$.

The complex scalar functions

$$\phi_L(\overset{1}{o}{}^A;\overset{1}{x})=\overset{1}{o}{}^A\overset{1}{o}{}^B\cdots\overset{1}{o}{}^C\phi_{AB\cdots C}(\overset{1}{x}) \tag{2.2.4}$$

and

$$\theta_R(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})=\underset{1}{\bar{o}}{}^{A'}\underset{1}{\bar{o}}{}^{B'}\cdots\underset{1}{\bar{o}}{}^{C'}\theta_{A'B'\cdots C'}(\overset{1}{x}) \tag{2.2.5}$$

define, respectively, the $\{-2s, 0; 0, 0\}$-left-handed and $\{0, 0; 0, 2s\}$-right-handed NID for the spinor fields on C_0^+ at $\overset{1}{x}{}^{AA'}$ (see, for instance, Penrose, 1980). We thus have the following NIDSs:

$$\text{LNIDS} = \{\phi_L(\overset{1}{o}{}^A; \overset{1}{x})\}, \qquad \text{RNIDS} = \{\theta_R(\underset{1}{\bar{o}}{}^{A'}; \overset{1}{x})\} \tag{2.2.6}$$

The spin scalar operators on C_0^+ which act on these NIDs are

$$\underset{1}{\hat{\pi}}_{s-} = \frac{\overset{1}{r}}{(\overset{1}{z})^{-2s}} \{\overset{1}{\mathbb{D}} - (1-2s)\overset{0}{\rho}(\overset{1}{x})\} \tag{2.2.7}$$

$$\underset{1}{\hat{\pi}}_{s+} = \frac{\overset{1}{r}}{(\underset{1}{\bar{z}})^{2s}} \{\overset{1}{\mathbb{D}} - (1+2s)\underset{0}{R}(\overset{1}{x})\} \tag{2.2.8}$$

We observe that $\underset{1}{\hat{\pi}}_{s-}$ is the complex conjugate of $\underset{1}{\hat{\pi}}_{s+}$. The operator (2.2.7) is the $\{2s, 2s; 0, 0\}$-left-handed spin scalar operator on C_0^+, whereas the operator (2.2.8) is the $\{0, 0; -2s, -2s\}$-right-handed spin scalar operator on C_0^+. In these defining expressions, $\overset{1}{r}$ is a suitable affine parameter along the generator γ_1 of C_0^+ and $\overset{1}{\mathbb{D}}$ is the differentiation operator in the direction of γ_1 at $\overset{1}{x}{}^{AA'}$. The scalar functions $\overset{0}{\rho}(\overset{1}{x})$, $\underset{0}{R}(\overset{1}{x})$ are real, and satisfy the reality relation on C_0^+

$$\overset{0}{\rho}(\overset{1}{x}) = \underset{0}{R}(\overset{1}{x}) \tag{2.2.9}$$

In fact, the real function $\overset{0}{\rho}(\overset{1}{x})$ is the convergence of the generators of C_0^+ at $\overset{1}{x}{}^{AA'}$ (see Penrose, 1980). Notice that (2.2.7), (2.2.8) can be rewritten as

$$\underset{1}{\hat{\pi}}_{s-} = \frac{\overset{1}{r}}{(\overset{1}{z})^{-2s}} \not{\hbar}_L, \qquad \underset{1}{\hat{\pi}}_{s+} = \frac{\overset{1}{r}}{(\underset{1}{\bar{z}})^{2s}} \not{\hbar}_R \tag{2.2.10}$$

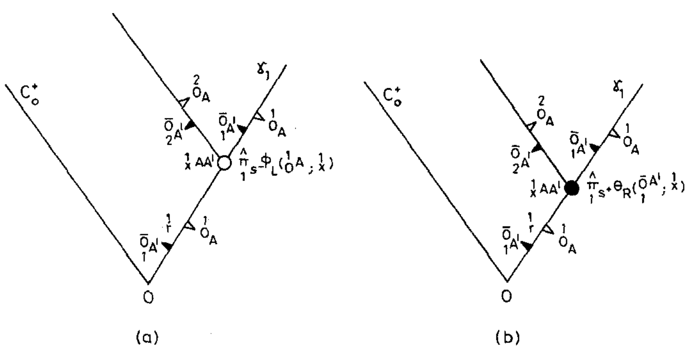

Fig. 1. A pair of spin bases $\{\{\overset{1}{o}_A, \overset{2}{o}_A\}, \{\underset{1}{\bar{o}}_{A'}, \underset{2}{\bar{o}}_{A'}\}\}$ is set up at a nonsingular point $\overset{1}{x}{}^{AA'}$ that lies on the future null cone C_0^+ of some origin O of $\mathbb{RM}$. The spinors $\overset{1}{o}_A$ and $\underset{1}{\bar{o}}_{A'}$ are chosen to be covariantly constant along the generator γ_1 of C_0^+ that passes through $\overset{1}{x}{}^{AA'}$ such that $\overset{1}{x}{}^{AA'}$ can be defined with respect to O in terms of a suitable affine parameter $\overset{1}{r}$: (a) the left-handed $\hat{\pi}$-null initial datum is specified at $\overset{1}{x}{}^{AA'}$, and represented by a white datum spot; (b) the right-handed $\hat{\pi}$-null initial datum is specified at $\overset{1}{x}{}^{AA'}$, and represented by a black datum spot.

where the $\not{p}$-operators are the usual real conformal-invariant form of the compact spin-coefficient derivative operators (see Penrose and Rindler, 1984). It is worth remarking that these operators are of the type $\{1, 0; 0, 1\}$.

We now let the operators (2.2.10) act on the NID (2.2.4) and (2.2.5), to obtain the $\hat{\pi}$-NIDSs on C_0^+

$$\hat{\pi}\text{-LNIDS} = \{\underset{1}{\hat{\pi}}_{s-}\phi_L(\overset{1}{o}{}^A;\overset{1}{x})\}, \qquad \hat{\pi}\text{-RNIDS} = \{\underset{1}{\hat{\pi}}_{s+}\theta_R(\underset{1}{\bar{o}}{}^{A'};\overset{1}{x})\} \quad (2.2.11)$$

A diagram illustrating the construction given here is shown in Figure 1.

2.3. The KAP Field Integrals

In order to introduce the explicit KAP field integrals, let us now define a point $\overset{2}{x}{}^{AA'}$ lying in the interior of the future cone of O by

$$\overset{2}{x}{}^{AA'} = \overset{1}{x}{}^{AA'} + \overset{2}{r}\overset{2}{o}{}^A \underset{2}{\bar{o}}{}^{A'} \quad (2.3.1a)$$

with $\overset{2}{r}$ being a suitable affine parameter along the null geodesic γ_2 of $\mathbb{RM}$ that passes through $\overset{1}{x}{}^{AA'}$ and $\overset{2}{x}{}^{AA'}$ (see Figure 2). We notice that $\overset{2}{x}{}^{AA'}$ is effectively future-null-separated from $\overset{1}{x}{}^{AA'}$. Indeed, we can reexpress (2.3.1a)

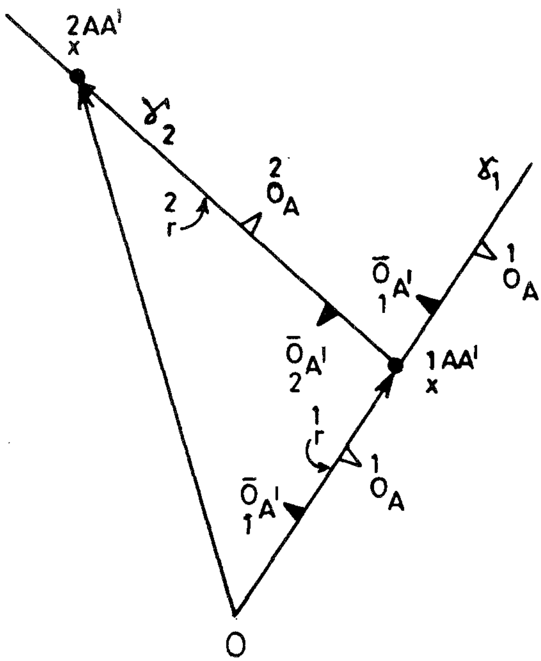

Fig. 2. The future-timelike vector representing a point $\overset{2}{x}{}^{AA'}$ that lies in the interior of the future cone of an origin O of $\mathbb{RM}$. The point $\overset{2}{x}{}^{AA'}$ is future-null-separated from $\overset{1}{x}{}^{AA'} \in C_0^+$.

as

$$\overset{2}{x}^{AA'} = \overset{1}{r}\overset{1}{o}^{A}\underset{1}{\bar{o}}^{A'} + \overset{2}{r}\overset{2}{o}^{A}\underset{2}{\bar{o}}^{A'} \tag{2.3.1b}$$

where $\overset{1}{r}$ is the affine parameter involved in the definitions (2.2.7) and (2.2.8).

The explicit KAP integral expressions for the fields are written as follows (see Penrose, 1963, 1975, 1980):

$$\phi_{AB\cdots C}(\overset{2}{x}) = \frac{1}{2\pi}\int_{C_2^- \cap C_0^+} \overset{2}{o}_A \overset{2}{o}_B \cdots \overset{2}{o}_C \frac{\not{k}_L \phi_L(\overset{1}{o}^M; \overset{1}{x})}{\overset{2}{r}} \overset{1}{\underset{\sim}{S}} \tag{2.3.2}$$

and

$$\theta^{A'B'\cdots C'}(\overset{2}{x}) = \frac{1}{2\pi}\int_{C_2^- \cap C_0^+} \underset{2}{\bar{o}}^{A'} \underset{2}{\bar{o}}^{B'} \cdots \underset{2}{\bar{o}}^{C'} \frac{\not{k}_R \theta_R(\underset{1}{\bar{o}}^{M'}; \overset{1}{x})}{\overset{2}{r}} \overset{1}{\underset{\sim}{S}} \tag{2.3.3}$$

where the meaning of $\overset{1}{\underset{\sim}{S}}$ will become clear shortly. According to these expressions, the massless free fields are determined at $\overset{2}{x}^{AA'}$ by their $\not{k}$-null initial data defined on C_0^+ at $\overset{1}{x}^{AA'}$. Upon evaluating each field, one performs an integral which is taken over the (spacelike) two-dimensional intersection of C_0^+ with the past null cone C_2^- of $\overset{2}{x}^{AA'}$ (see Figure 3). In fact, we have (see Section 4.14 of Penrose and Rindler, 1984)

$$C_2^- \cap C_0^+ \cong S^2 \tag{2.3.4}$$

Thus, $\overset{1}{\underset{\sim}{S}}$ is a two-form that defines the element of surface area of S^2 at $\overset{1}{x}^{AA'}$. We can, therefore, formally reexpress each of the above integral expressions as a field integral which is taken over a spacelike two-sphere. It must be

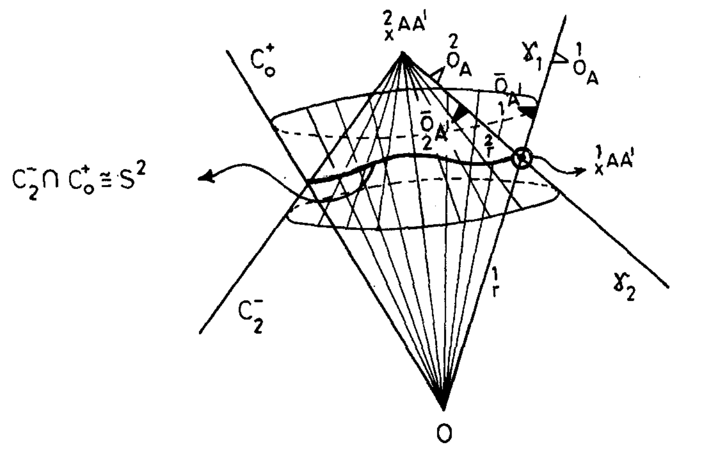

Fig. 3. The geometry for the Kirchoff–D'Adhemar–Penrose integral expressions. Each field integral is taken over the intersection of C_0^+ with the past null cone C_2^- of $\overset{2}{x}^{AA'}$. The spinor fields are determined at $\overset{2}{x}^{AA'}$ from their $\not{k}$-null initial data specified at $\overset{1}{x}^{AA'}$.

remarked that these integrals are not invariant under arbitrary scalings of the elements of the spin bases introduced in Section 2.2.

2.4. The Invariant Field Integrals

We shall next introduce a manifestly scaling invariant (SI) integral expression for each of the massless free fields, based upon a particularly simple Huyghens' principle. It will be seen also that the SI field integrals can be rewritten in a remarkably simple form in terms of certain field densities on $\underset{1}{C}_0^+$.

Let $\overset{1}{\mathbb{K}}$ denote the two-dimensional abstract space of pairs of null geodesic segments of $\mathbb{RM}$ incident at $\overset{1}{x}^{AA'}$ such that, for each pair, one segment starts at the origin O of $\mathbb{RM}$ and terminates at $\overset{1}{x}^{AA'}_{1}$, while the other starts at $\overset{1}{x}^{AA'}$ and terminates at $\overset{2}{x}^{AA'}$. Any element of $\overset{1}{\mathbb{K}}$ can be seen as a null zigzag in $\mathbb{RM}$ whose edges are defined according to the above prescription.

We now define an SI two-form $\overset{1}{\underset{\sim}{K}}$ on $\overset{1}{\mathbb{K}}$ as follows:

$$\overset{1}{\underset{\sim}{K}} = \overset{1}{\underset{\sim}{S}}/\overset{1}{z}\underset{1}{\bar{z}}\overset{12}{rr} \qquad (2.4.1)$$

where $\overset{1}{z}$, $\underset{1}{\bar{z}}$ are the inner products at $\overset{1}{x}^{AA'}$ defined in (2.2.3), and $\overset{1}{r}$, $\overset{2}{r}$ are the affine parameters involved in the relation (2.3.1b). It becomes evident that

$$\overset{1}{\mathbb{K}} \cong S^2 \qquad (2.4.2)$$

In our terminology, we can state Huyghens' principle as follows:

A spinning massless free field on $\mathbb{RM}$ is entirely determined at some fixed point $\overset{2}{x}^{AA'}$ that lies in the interior of the future cone of O and that is future-null-separated from an arbitrary nonsingular point $\overset{1}{x}^{AA'}$ of C_0^+ by its $\hat{\pi}$-null initial datum on C_0^+ at $\overset{1}{x}^{AA'}$.

For the left-handed field, this formulation of Huyghens' principle is explicitly exhibited by the following SI left-handed field integral:

$$\phi_{AB\cdots C}(\overset{2}{x}) = \frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \overset{2}{o}_A\overset{2}{o}_B\cdots\overset{2}{o}_C\underset{1}{\hat{\pi}}_{s-}\phi_L(\overset{1}{o}^M;\overset{1}{x})\overset{1}{\underset{\sim}{K}} \qquad (2.4.3)$$

Similarly, for the right-handed field, the SI field integral is

$$\theta^{A'B'\cdots C'}(\overset{2}{x}) = \frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \underset{2}{\bar{o}}^{A'}\underset{2}{\bar{o}}^{B'}\cdots\underset{2}{\bar{o}}^{C'}\underset{1}{\hat{\pi}}_{s+}\theta_R(\underset{1}{\bar{o}}^{M'};\overset{1}{x})\overset{1}{\underset{\sim}{K}} \qquad (2.4.4)$$

A remarkably simple form of the field integrals (2.4.3) and (2.4.4) can be achieved by defining the following SI massless free-field densities on C_0^+ at $\overset{1}{x}^{AA'}$:

$$\Phi_{AB\cdots C}(\overset{1}{x}) = \overset{2}{o}_A\overset{2}{o}_B\cdots\overset{2}{o}_C\underset{1}{\hat{\pi}}_{s-}\phi_L(\overset{1}{o}^M;\overset{1}{x}) \qquad (2.4.5)$$

and

$$\Theta^{A'B'\cdots C'}(\overset{1}{x}) = \underset{2}{\bar{o}}^{A'}\underset{2}{\bar{o}}^{B'} \cdots \underset{2}{\bar{o}}^{C'}\underset{1}{\hat{\pi}}_{s+}\theta_R(\underset{1}{\bar{o}}^{M'}; \overset{1}{x}) \tag{2.4.6}$$

These yield

$$\phi_{AB\cdots C}(\overset{2}{x}) = \frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \Phi_{AB\cdots C}(\overset{1}{x})\overset{1}{\underset{\sim}{K}} \tag{2.4.7}$$

and

$$\theta^{A'B'\cdots C'}(\overset{2}{x}) = \frac{1}{2\pi}\int_{\overset{1}{\mathbb{K}}} \Theta^{A'B'\cdots C'}(\overset{1}{x})\overset{1}{\underset{\sim}{K}} \tag{2.4.8}$$

It will be seen in Section 3 that the simplicity of these integrals is preserved when we translate them into twistorial terms.

3. TWISTORIAL TRANSCRIPTION

For transcribing the SI field integrals into twistor form, we shall make use of the geometric pictures of Section 2. It will be seen that the choice of C_0^+ as the NID hypersurface for the spinor fields allows one to carry out the twistorial transcription in a direct way.

3.1. Twistor Null Initial Data

We define the null dual twistors through $\overset{1}{x}^{AA'}$

$$\overset{1}{W}_\alpha = (\overset{1}{o}_A, -i\overset{1}{x}^{AA'}\overset{1}{o}_A) = (\overset{1}{o}_A, \overset{1}{W}^{A'}) \tag{3.1.1a}$$

$$\overset{2}{W}_\alpha = (\overset{2}{o}_A, -i\overset{1}{x}^{AA'}\overset{2}{o}_A) = (\overset{2}{o}_A, \overset{2}{W}^{A'}) \tag{3.1.1b}$$

Notice that these dual twistors are associated with the null geodesics γ_1 and γ_2, respectively. Their complex conjugates are given by

$$\underset{1}{Z}^\beta = \underset{1}{\bar{W}}^\beta = (i\overset{1}{x}^{AA'}\underset{1}{\bar{o}}_{A'}, \underset{1}{\bar{o}}_{A'}) = (\underset{1}{Z}^A, \underset{1}{\bar{o}}_{A'}) \tag{3.1.2a}$$

$$\underset{2}{Z}^\beta = \underset{2}{\bar{W}}^\beta = (i\overset{1}{x}^{AA'}\underset{2}{\bar{o}}_{A'}, \underset{2}{\bar{o}}_{A'}) = (\underset{2}{Z}^A, \underset{2}{\bar{o}}_{A'}) \tag{3.1.2b}$$

It is evident that the above null twistors intersect at $\overset{1}{x}^{AA'}$, namely

$$\underset{2}{Z}^\lambda \overset{1}{W}_\lambda = 0 = \underset{1}{Z}^\lambda \overset{2}{W}_\lambda \quad \text{at} \quad \overset{1}{x}^{AA'} \tag{3.1.3}$$

Let $\mathbb{PN}^*$ denote the null portion of the dual projective twistor space. The dual twistors $\overset{1}{W}_\alpha$ and $\overset{2}{W}_\alpha$ are defined in $\mathbb{PN}^*$, respectively, as the intersections of the dual projective line $\overset{1}{X}^*$ representing $\overset{1}{x}^{AA'}$ with the fixed dual projective lines $\underset{0}{X}^*$ and $\underset{2}{X}^*$ representing the fixed points O and $\overset{2}{x}^{AA'}$. Each of these fixed dual projective lines is topologically a Riemann sphere S^2 (see Penrose and Rindler, 1986) whose points are associated with all the dual twistors incident with the point of $\mathbb{RM}$ corresponding to it. It follows that, as $\overset{1}{x}{}^{AA'}$ varies suitably, two subspaces $\mathscr{D}_0^*$ and $\mathscr{D}_2^*$ of the Riemann spheres arise, $\mathscr{D}_0^*$ consisting only of points associated with $\overset{1}{W}_\alpha$, and $\mathscr{D}_2^*$ consisting only of points associated with $\overset{2}{W}_\alpha$. Evidently, $\mathscr{D}_0^*$ is contained in the Riemann sphere $\mathscr{R}_0^*$ representing O, while $\mathscr{D}_2^*$ is contained in the Riemann sphere $\mathscr{R}_2^*$ representing $\overset{2}{x}{}^{AA'}$.

We now define the twistor $\{-2s, 0; 0, 0\}$-left-handed NI datum on the product space $\mathscr{D}^*$ of $\mathscr{D}_0^*$ with $\mathscr{D}_2^*$ as

$$(I^{\mu\nu}\overset{1}{W}_\mu \overset{2}{W}_\nu)^{-2s+1}\Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{3.1.4}$$

where $\Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ is homogeneous of degree -1 in $\overset{1}{W}_\alpha$ and $2s-1$ in $\overset{2}{W}_\alpha$, and satisfies

$$\overset{1}{W}_\lambda \frac{\partial}{\partial \overset{2}{W}_\lambda} \Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = 0 \tag{3.1.5}$$

Thus, the $\overset{2}{W}_\alpha$ dependence of $\Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ only comes through $\overset{1}{W}_{[\mu} \overset{2}{W}_{\nu]}$ [see (3.4.5) below].

The twistor NI datum corresponding to (2.2.5) is defined similarly. All the relevant projective lines lie in the null portion $\mathbb{PN}$ of the projective twistor space. The two subspaces $\mathscr{D}_0$ and $\mathscr{D}_2$ of the relevant Riemann spheres $\mathscr{R}_0$ and $\mathscr{R}_2$ representing O and $\overset{2}{x}{}^{AA'}$ here consist only of points associated with $\underset{1}{Z}{}^\beta$ and $\underset{2}{Z}{}^\beta$, respectively. We thus have

$$(I_{\mu\nu}\underset{1}{Z}{}^\mu \underset{2}{Z}{}^\nu)^{2s+1}\Theta_R(\underset{1}{Z}{}^\beta, \underset{2}{Z}{}^\beta) \tag{3.1.6}$$

as the definition of the twistor $\{0, 0; 0, 2s\}$-right-handed NI datum on the product space $\mathscr{D}$ of $\mathscr{D}_0$ with $\mathscr{D}_2$. The twistor function $\Theta_R(\underset{1}{Z}{}^\beta, \underset{2}{Z}{}^\beta)$ satisfies on $\mathscr{D}$ properties similar to the previous ones.

Each of the twistor functions $\Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ and $\Theta_R(\underset{1}{Z}{}^\beta, \underset{2}{Z}{}^\beta)$ is a holomorphic twistor function in both twistors on the appropriate product space, having a suitable singularity set (see Section 3.4 below). The standard twistor functions representing the spinning massless free fields will emerge as contour integrals of holomorphic one-forms involving certain derivatives of these functions (see Section 4). These integrals are taken over one-real-dimensional closed contours that cannot be continuously shrunk to a point without crossing some singularity.

3.2. Twistorial Expressions for $\hat{\pi}$-Operators and Twistor $\hat{\pi}$-NID

We now derive simple twistorial expressions for the spin scalar operators $\underset{1}{\hat{\pi}}_{s-}$ and $\underset{1}{\hat{\pi}}_{s+}$. These twistorial operators act on the appropriate twistor-function kernels of the twistor NID (3.1.4) and (3.1.6), leading us thereby to the holomorphic twistor $\hat{\pi}$-NID on $\mathscr{D}^*$ and $\mathscr{D}$.

Recall that the differentiation operator at $\overset{1}{x}{}^{AA'}$ in the direction of the generator γ_1 of C_0^+ is given by

$$\overset{1}{\mathbb{D}} = \overset{1}{o}{}^{A}\underset{1}{\bar{o}}{}^{A'}\overset{1}{\nabla}_{AA'} \tag{3.2.1}$$

Using (3.1.1b) together with the relation on C_0^+

$$\overset{1}{x}{}^{AA'} = \overset{1}{r}\overset{1}{o}{}^{A}\underset{1}{\bar{o}}{}^{A'} \tag{3.2.2}$$

we readily obtain

$$\overset{1}{\mathbb{D}} = \frac{1}{\overset{1}{r}}\overset{2}{W}{}^{A'}\frac{\partial}{\partial \overset{2}{W}{}^{A'}} \tag{3.2.3}$$

Also, combining (3.2.2) with the defining expression for the convergence of the generators of C_0^+ at $\overset{1}{x}{}^{AA'}$

$$\overset{0}{\rho}(\overset{1}{x}) = (\overset{1}{o}{}^{B}\overset{2}{o}_{B})^{-1}\overset{1}{o}_{A}\overset{2}{o}{}^{C}\underset{1}{\bar{o}}{}^{C'}\overset{1}{\nabla}_{CC'}\overset{1}{o}{}^{A} \tag{3.2.4}$$

we find that

$$\overset{0}{\rho}(\overset{1}{x}) = -\frac{1}{\overset{1}{r}} \tag{3.2.5}$$

In fact, the twistorial relation

$$\overset{1}{z} = I^{\mu\nu}\overset{2}{W}_{\mu}\overset{1}{W}_{\nu} \tag{3.2.6}$$

holds at $\overset{1}{x}{}^{AA'}$ (see, for example, Penrose and Rindler, 1986). Using (3.2.3), (3.2.5), and (3.2.6), together with the homogeneity properties and the definition (2.2.7), we conclude that the $\hat{\pi}$-operator acting upon $\Phi_L(\overset{1}{W}_{\alpha}, \overset{2}{W}_{\alpha})$ is expressed by

$$\underset{1}{\hat{\pi}}_{s-} = \frac{(-1)^{-2s+1}}{(I^{\mu\nu}\overset{1}{W}_{\mu}\overset{2}{W}_{\nu})^{-2s}}\overset{2}{o}_{A}\frac{\partial}{\partial \overset{2}{o}_{A}} \tag{3.2.7}$$

Taking the complex conjugate of (3.2.7), we obtain

$$\underset{1}{\hat{\pi}}_{s+} = \frac{(-1)^{2s+1}}{(I_{\mu\nu}\underset{1}{Z}{}^{\mu}\underset{2}{Z}{}^{\nu})^{2s}}\underset{2}{\bar{o}}_{A'}\frac{\partial}{\partial \underset{2}{\bar{o}}_{A'}} \tag{3.2.8}$$

which is the corresponding twistorial expression for the $\{0, 0; -2s, -2s\}$-right-handed spin scalar operator that acts on $\Theta_R(\underset{1}{Z}{}^{\beta}, \underset{2}{Z}{}^{\beta})$. Note that the corresponding twistorial expressions for the conformal $\not{p}$-operators can be derived by inserting (3.2.7) and (3.2.8) into (2.2.10), taking into account (3.2.6) and its complex conjugate.

It now becomes clear that if we let the twistorial operators (3.2.7) and (3.2.8) act appropriately on the twistor functions to which we have referred above, we are led to

$$(I^{\mu\nu}\overset{1}{W}_\mu\overset{2}{W}_\nu)^{-2s+1}\underset{1}{\hat{\pi}}_{s-}\Phi_L(\overset{1}{W}_\alpha,\overset{2}{W}_\alpha)$$

$$=(-1)^{-2s+1}(I^{\mu\nu}\overset{1}{W}_\mu\overset{2}{W}_\nu)\overset{2}{o}_A\frac{\partial}{\partial\overset{2}{o}_A}\Phi_L(\overset{1}{W}_\alpha,\overset{2}{W}_\alpha) \qquad (3.2.9)$$

and

$$(I_{\mu\nu}\underset{1}{Z}^\mu\underset{2}{Z}^\nu)^{2s+1}\underset{1}{\hat{\pi}}_{s+}\Theta_R(\underset{1}{Z}^\beta,\underset{2}{Z}^\beta)$$

$$=(-1)^{2s+1}(I_{\mu\nu}\underset{1}{Z}^\mu\underset{2}{Z}^\nu)\underset{2}{\bar{o}}_{A'}\frac{\partial}{\partial\underset{2}{\bar{o}}_{A'}}\Theta_R(\underset{1}{Z}^\beta,\underset{2}{Z}^\beta) \qquad (3.2.10)$$

The twistor functions (3.2.9) and (3.2.10) are the holomorphic twistor $\hat{\pi}$-NID on $\mathscr{D}^*$ and $\mathscr{D}$, respectively. These twistor NID are indeed associated with the elements of the sets $\hat{\pi}$-LNIDS and $\hat{\pi}$-RNIDS introduced earlier [see (2.2.11)].

3.3. Holomorphic Twistorial $\overset{1}{\underset{\sim}{K}}$-Forms

It has been seen that the remarkably simple SI field integrals for the spinning massless free fields explicitly involve an SI two-form $\overset{1}{\underset{\sim}{K}}$ defined on the two-dimensional space of suitable null zigzags in $\mathbb{RM}$ [see (2.4.7) and (2.4.8)]. In order to write down our twistorial field integrals (see Section 3.4), we need to translate $\overset{1}{\underset{\sim}{K}}$ into twistorial terms. The twistorial expressions arising here are indeed associated with null zigzags in $\mathbb{RM}$ whose edge-sets, at each stage, possess two edges. Generalized holomorphic expressions in the case of an arbitrary number of edges have been given in Cardoso (1988). Actually, two conjugate holomorphic expressions will emerge here, one on $\mathscr{D}^*$ involving only the dual twistors $\overset{1}{W}_\alpha$, $\overset{2}{W}_\alpha$, and the other on $\mathscr{D}$ involving only the twistors $\underset{1}{Z}^\beta$, $\underset{2}{Z}^\beta$.

Consider the two-form element of surface area $\overset{1}{\underset{\sim}{S}}$ of the spacelike two-sphere $\underset{1}{S}^2$ at $\overset{1}{x}^{AA'}$. It is evident that this two-form enters into the definition of $\overset{1}{\underset{\sim}{K}}$ via (2.4.1). Its explicit defining expression at $\overset{1}{x}^{AA'}$ is

$$\overset{1}{\underset{\sim}{S}}=\frac{i}{\underset{1}{\overset{1}{z}\bar{z}}}\overset{1}{o}_A\underset{2}{\bar{o}}_{A'}\,d\overset{1}{x}^{AA'}\wedge\overset{2}{o}_B\underset{1}{\bar{o}}_{B'}\,d\overset{1}{x}^{BB'} \qquad (3.3.1)$$

By differentiating out the twistor relations

$$\overset{1}{W}^{A'}=-i\overset{1}{x}^{AA'}\overset{1}{o}_A,\qquad \overset{2}{W}^{A'}=-i\overset{1}{x}^{AA'}\overset{2}{o}_A \qquad (3.3.2)$$

and making suitable contractions, we arrive at two relations which, when substituted together with (3.2.6) and its complex conjugate into (3.3.1), yield the SI expression

$$\underset{\sim}{\overset{1}{S}} = (-i)\frac{\underset{2}{Z}^{\lambda}\, d\overset{1}{W}_{\lambda} \wedge \underset{1}{Z}^{\tau}\, d\overset{2}{W}_{\tau}}{(I^{\mu\nu}\overset{1}{W}_{\mu}\overset{2}{W}_{\nu})(I_{\rho\sigma}\underset{1}{Z}^{\rho}\underset{2}{Z}^{\sigma})} \tag{3.3.3}$$

The actual evaluation of the spinor fields involves also holding both the origin O and $\overset{2}{x}^{AA'}$ fixed. This procedure enables us to derive after a short calculation

$$\underset{1}{Z}^{\mu}\, d\overset{2}{W}_{\mu} = i\overset{2}{r}(I_{\mu\nu}\underset{1}{Z}^{\mu}\underset{2}{Z}^{\nu})(I^{\lambda\tau}\overset{2}{W}_{\lambda}\, d\overset{2}{W}_{\tau}) \tag{3.3.4}$$

and

$$\underset{2}{Z}^{\mu}\, d\overset{1}{W}_{\mu} = i\overset{1}{r}(I_{\mu\nu}\underset{1}{Z}^{\mu}\underset{2}{Z}^{\nu})(I^{\lambda\tau}\overset{1}{W}_{\lambda}\, d\overset{1}{W}_{\tau}) \tag{3.3.5}$$

Now, combining (3.3.4) and (3.3.5) with (3.3.3), making use of (3.2.6) and its complex conjugate once again, we obtain the following SI twistorial expression on $\mathscr{D}^*$:

$$\underset{\sim}{\overset{1}{K}} = i\frac{I^{\mu\nu}\overset{1}{W}_{\mu}\, d\overset{1}{W}_{\nu} \wedge I^{\lambda\tau}\overset{2}{W}_{\lambda}\, d\overset{2}{W}_{\tau}}{(I^{\alpha\beta}\overset{1}{W}_{\alpha}\overset{2}{W}_{\beta})^{2}} \tag{3.3.6}$$

Clearly, the complex conjugate of (3.3.6) is the twistorial expression on $\mathscr{D}$ given by

$$\underset{\sim}{\overset{1}{K}} = (-i)\frac{I_{\mu\nu}\underset{1}{Z}^{\mu}\, d\underset{1}{Z}^{\nu} \wedge I_{\lambda\tau}\underset{2}{Z}^{\lambda}\, d\underset{2}{Z}^{\tau}}{(I_{\alpha\beta}\underset{1}{Z}^{\alpha}\underset{2}{Z}^{\beta})^{2}} \tag{3.3.7}$$

The formulas (3.3.6) and (3.3.7) are the desired SI holomorphic expressions for $\underset{\sim}{\overset{1}{K}}$.

3.4. The SI Twistorial Field Integrals

The holomorphic expressions for $\underset{\sim}{\overset{1}{K}}$ together with the holomorphic twistor $\hat{\pi}$-NID of Section 3.2 lead us to particularly simple SI twistorial integral expressions for the spinning massless free fields. Each spinor field is then expressed in terms of an integral involving only an SI holomorphic twistor two-form on the appropriate product space.

It is easy to verify that, as $\overset{1}{x}^{AA'}$ varies suitably, the following relation arises:

$$\overset{2}{o}_{A} I^{\alpha\beta}\overset{1}{W}_{\alpha}\, d\overset{1}{W}_{\beta} = (I^{\alpha\beta}\overset{1}{W}_{\alpha}\overset{2}{W}_{\beta})\, d\overset{1}{o}_{A} + \underset{\sim}{\nu}\overset{1}{o}_{A} \tag{3.4.1}$$

where $\underset{\sim}{\nu}$ is a one-form of the type $\{1, 1; 0, 0\}$. By replacing (3.2.9) and (3.3.6) into (2.4.3), taking both (3.4.1) and (3.1.5) into account, we obtain the SI twistorial field integral

$$\phi_{AB\cdots C}(\overset{2}{x}) = \frac{(-1)^{-2s+1}}{2\pi i} \oint_{S^1\times S^1} \overset{2}{o}_A \overset{2}{o}_B \cdots \overset{2}{o}_C \underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \wedge I^{\mu\nu} \overset{2}{W}_\mu \, d\overset{2}{W}_\nu \tag{3.4.2}$$

where $\underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha)$ is the holomorphic twistor one-form on $\mathscr{D}^*$,

$$-\frac{\partial}{\partial \overset{2}{W}_\lambda} \Phi_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \, d\overset{1}{W}_\lambda \tag{3.4.3}$$

The product $S^1 \times S^1$ appearing beneath the integral sign of the formal expression (3.4.2) means that one has first to perform the $\overset{1}{W}$ integral along a closed path γ_L ($\cong S^1$) lying in $\mathscr{D}_0^* \subset \mathscr{R}_0^*$, and then carry out the $\overset{2}{W}$ integral along a closed path Γ_L ($\cong S^1$) lying in $\mathscr{D}_2^* \subset \mathscr{R}_2^*$ [see (4.1) and (4.5) below]. At each stage, the contour suitably avoids all the singularities of the relevant integrand. The entire singularity set appears, then, to be given as the union of two separated (closed) subsets of the appropriate Riemann sphere. It is evident that (3.4.3) satisfies the properties

$$\overset{2}{W}_\lambda \frac{\partial}{\partial \overset{2}{W}_\lambda} \underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) = (2s-2)\underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{3.4.4}$$

and

$$\underset{\sim}{\Phi}_L(\mu \overset{1}{W}_\alpha, \lambda \overset{1}{W}_\alpha + \nu \overset{2}{W}_\alpha) = \nu^{2s-2} \underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{3.4.5}$$

with $\mu, \nu \in \mathbb{C} - \{0\}$, and $\lambda \in \mathbb{C}$.

A procedure similar to the one which led to (3.4.2) can now be adopted to obtain the SI twistorial field integral expression for the right-handed spinnning massless free field $\theta^{A'B'\cdots C'}(\overset{2}{x})$. Thus, the resulting formal expression is

$$\theta^{A'B'\cdots C'}(\overset{2}{x}) = \frac{(-1)^{2s+1}}{2\pi i} \oint_{S^1\times S^1} \underset{2}{\bar{o}}^{A'} \underset{2}{\bar{o}}^{B'} \cdots \underset{2}{\bar{o}}^{C'} \underset{\sim}{\Theta}_R(\underset{1}{Z}^\beta, \underset{2}{Z}^\beta) \wedge I_{\mu\nu} \underset{2}{Z}^\mu \, d\underset{2}{Z}^\nu \tag{3.4.6}$$

where $\underset{\sim}{\Theta}_R(\underset{1}{Z}^\beta, \underset{2}{Z}^\beta)$ is the holomorphic twistor one-form on $\mathscr{D}$,

$$\frac{\partial}{\partial \underset{2}{Z}^\lambda} \Theta_R(\underset{1}{Z}^\beta, \underset{2}{Z}^\beta) \, d\underset{1}{Z}^\lambda \tag{3.4.7}$$

and the relevant contours γ_R and Γ_R lie now in $\mathscr{D}_0 \subset \mathscr{R}_0$ and $\mathscr{D}_2 \subset \mathscr{R}_2$, respectively. Clearly, the one-form (3.4.7) satisfies properties similar to (3.4.4) and (3.4.5).

The prescription leading to the twistorial integral (3.4.2) is illustrated in Figure 4. A projective picture associated with the twistorial integral (3.4.6) can be similarly drawn.

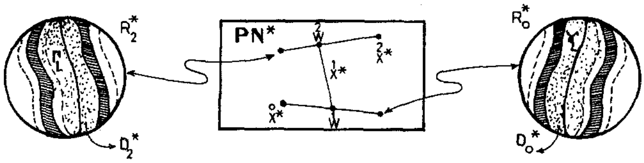

Fig. 4. The dual projective lines $\overset{0}{X}{}^*$, $\overset{1}{X}{}^*$, and $\overset{2}{X}{}^*$ in $\mathbb{PN}^*$ representing the points O, $\overset{1}{x}{}^{AA'}$, and $\overset{2}{x}{}^{AA'}$ in $\mathbb{RM}$. The line $\overset{1}{X}{}^*$ meets both $\overset{0}{X}{}^*$ and $\overset{2}{X}{}^*$ at $\overset{1}{W}_\alpha$ and $\overset{2}{W}_\alpha$, respectively, provided that $\overset{1}{x}{}^{AA'}$ is null-separated from both O and $\overset{2}{x}{}^{AA'}$. Each of the fixed projective lines $\overset{0}{X}{}^*$ and $\overset{2}{X}{}^*$ is (topologically) represented by a Riemann sphere S^2. As $\overset{1}{x}{}^{AA'}$ varies suitably, two subspaces $\mathscr{D}_0^*$ and $\mathscr{D}_2^*$ arise, $\mathscr{D}_0^*$ being contained in the Riemann sphere $\mathscr{R}_0^*$ associated with O and $\mathscr{D}_2^*$ being contained in the Riemann sphere $\mathscr{R}_2^*$ associated with $\overset{2}{x}{}^{AA'}$. The $\overset{1}{W}$ integration is taken along a closed path γ_L lying in $\mathscr{D}_0^*$, while the $\overset{2}{W}$ integration is taken along a closed path Γ_L lying in $\mathscr{D}_2^*$. At each stage, the contour separates the entire singularity set of the relevant integrand into two disconnected closed sets.

4. THE UNIVERSAL PENROSE CONTOUR INTEGRAL FORMULAS

In this section, we shall see how the standard twistor functions representing the spinning massless free fields can be suitably given in terms of contour integrals of the holomorphic twistor one-forms that appear in the twistorial field integrals exhibited before. The universal Penrose contour integral formulas for these fields are then readily derived.

As was previously indicated, one has to perform two independent contour integrals whenever the actual evaluation of either of our twistorial field integrals is required. This fact enables one to integrate (3.4.3) and (3.4.7), giving rise to the following holomorphic functions:

$$f_L(\overset{2}{W}_\alpha) = (-1)^{-2s+1} \oint_{\gamma_L \subset \mathscr{D}_0^*} \underset{\sim}{\Phi}_L(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha) \tag{4.1}$$

defined on $\mathscr{D}_2^*$, and

$$f_R(\underset{2}{Z}^\beta) = (-1)^{2s+1} \oint_{\gamma_R \subset \mathscr{D}_0} \underset{\sim}{\Theta}_R(\underset{1}{Z}^\beta, \underset{2}{Z}^\beta) \tag{4.2}$$

defined on $\mathscr{D}_2$. These holomorphic functions are indeed homogeneous of degrees $2s-2$ and $-2s-2$ in $\overset{2}{W}_\alpha$ and $\underset{2}{Z}^\beta$, respectively. In the left-handed case, for example, the homogeneity property follows immediately from (3.1.4) and (3.4.3). Explicitly, we have

$$f_L(\nu \overset{2}{W}_\alpha) = \nu^{2s-2} f_L(\overset{2}{W}_\alpha) \tag{4.3}$$

and

$$f_R(\mu \underset{2}{Z}^\beta) = \mu^{-2s-2} f_R(\underset{2}{Z}^\beta) \tag{4.4}$$

with μ, $\nu \in \mathbb{C} - \{0\}$. Hence, the holomorphic functions (4.1) and (4.2) actually are the standard twistor functions which generate the fields. We are therefore again led to the universal Penrose contour integral formulas for the spinning massless free fields. Replacing (4.1) and (4.2) into (3.4.2) and (3.4.6), respectively, we obtain

$$\phi_{AB\cdots C}(\overset{2}{x}) = \frac{1}{2\pi i}\oint_{\Gamma_L \subset \mathscr{D}_2^*} \overset{2}{o}_A \overset{2}{o}_B \cdots \overset{2}{o}_C f_L(\overset{2}{W}_\alpha) I^{\mu\nu} \overset{2}{W}_\mu \, d\overset{2}{W}_\nu \tag{4.5}$$

and

$$\theta^{A'B'\cdots C'}(\overset{2}{x}) = \frac{1}{2\pi i}\oint_{\Gamma_R \subset \mathscr{D}_2} \underset{2}{\bar{o}}^{A'} \underset{2}{\bar{o}}^{B'} \cdots \underset{2}{\bar{o}}^{C'} f_R(\underset{2}{Z}^\beta) I_{\mu\nu} \underset{2}{Z}^\mu \, d\underset{2}{Z}^\nu \tag{4.6}$$

which are the formulas referred to above.

5. CONCLUDING REMARKS

In this paper, we have presented a twistorial transcription of (appropriately modified) KAP integral expressions. It was shown that the choice of the future null cone of the origin of $\mathbb{RM}$ as the NID hypersurface for the spinning massless free fields led us to the definition of the SI field densities which provide simple formal expressions for the spinor fields. These expressions not only enhance the Kirchoff-like character of the field integrals, but also allow them to be neatly fitted in with the twistor formalism. In addition, all the contours involved in our twistorial field integrals turn out to be defined in a remarkably simple way.

The most important feature of the simple holomorphic twistor structures exhibited here is the correspondence between the valence of the involved twistors and the handedness of the spinor fields. Only twistors lying in $\mathbb{PN}^*$ enter into the twistorial left-handed field integral, while only twistors lying in $\mathbb{PN}$ enter into the twistorial right-handed field integral. Another feature of our twistorial field integrals is the splitting of the contour over which each of the holomorphic twistor two-forms is to be integrated. The modified expressions for the spinning massless free fields in $\mathbb{RM}$ can be formally given as SI integrals taken over spheres S^2. After having transcribed these field integrals into twistorial terms, we found that each of the twistorial field integrals turns out to be taken over a (suitable) contour whose topology is $S^1 \times S^1$. If the spinor fields are to be interpreted as wave functions, these features will play an important role. It is worth remarking that the situation considered here is apparently very different from that considered by Bramson, Sparling, and Penrose in connection with the problem of finding an inverse to each of the standard contour integral expressions (see Penrose, 1975). In their work, the universal contour integral formulas for the spinor fields are given at the outset. It was stated that, in the left-handed case, the relevant twistor function can be formally defined in terms of a contour integral which is taken along an open path lying in the appropriate subspace. This contour integral actually involves the same holomorphic twistor one-form as the one obtained explicitly here [see (3.4.3)], but leads one to a twistor quantity depending upon two arbitrary holomorphic functions of the appropriate dual twistor.

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